Linear Algebra 1 Final Exam Solutions

1. Prove the Steinitz exchange lemma.

If $\{w_1, w_2, \dots, w_m\}$ is a set of m linearly independent vectors in V and $\{v_1, v_2, \dots, v_n\}$ spans V, then:

1. $m \leq n$.

2. After reordering the v_i the set $\{w_1, w_2, \dots, w_m, v_{m+1}, \dots, v_n\}$ spans V.

Solution: could be found on the internet.

2. Prove Cramer's rule.

Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows: $Ax = b$. Where $|A| \neq 0$, $x = (x_1, \dots, x_n)^t$ then:

$$x_i = \frac{\det(A_i)}{\det(A)}.$$

Where A_i is the matrix formed by replacing the *i*-th column of *A* by the column vector *b*.

Solution: could be found on the internet.

3. $U \overset{S}{\rightarrow} V \overset{T}{\rightarrow} W$, where S, T are linear transformations. Prove:

$T \circ S$ injection $\Leftrightarrow$ S injection and $Im\, S \cap Ker\, T = \{0\}$.

$\Rightarrow$ Let $u, u' \in U$ such that $S(u) = S(u')$ then $(T \circ S)(u) = (T \circ S)(u')$ but $T \circ S$ is injective hence $u = u'$ hence S is injective.

Let $v \in Im\, S \cap Ker\, T$ hence there exsits $u \in U$ such that $S(u) = v$ and $T(v) = 0$. Hence $(T \circ S)(u) = 0 = (T \circ S)(0)$ but $T \circ S$ is injective hence $u = 0$ therefore $v = S(0) = 0$, $0 \in Im\, S \cap Ker\, T$ hence $Im\, S \cap Ker\, T = \{0\}$.

$\Leftarrow$ Let $u \in Ker\, T \circ S$ then $(T \circ S)(u) = 0$ let $v = S(u)$ then $v \in Im\, S \cap Ker\, T$ hence $v = 0$ then $0 = S(u) = S(0)$, S is injective hence $u = 0$. Hence $Ker\, T \circ S = \{0\}$ hence $T \circ S$ is an injection.

4. Let $\mathbb{F}$ be a finite field with q elements. Find how many isomorphisms $\mathbb{F}^4 \rightarrow \mathbb{F}^4$ exists.

Solution:

It is well known that there exists a unique linear transformation $T: V \rightarrow V$ such that:

$T(v_1) = w_1$, $T(v_2) = w_2, \dots, T(v_n) = w_n$.

Where $\{v_1, \dots, v_n\}, \{w_1, \dots, w_n\}$ are bases of V.

Notice that T is Surjection (on-to), hence T injection and hence isomorphism.

Back to our problem , if we count how many bases $\mathbb{F}^4$ has , we are done.

Let us "build" a basis for $\mathbb{F}^4$, the first element could by anything but the zero vector thus we have $q^4 - 1$ choises. Now for the second element we have all the vectors expect the vectors that are spanned by the first vector that we chose , there $q^4 - q$ vectors like that. For the third element we $q^4 - q^2$ vectors that we can choose, for the last element we $q^4 - q^3$ choices. Thus we have $(q^4 - 1)(q^4 - q)(q^4 - q^2)(q^4 - q^3)$ isomorphisms $\mathbb{F}^4 \to \mathbb{F}^4$.

5. Let $A, B \in M_n$ such that $rank\ AB = rank\ B$. Prove that every solution to $(AB)X = 0$ is a solution for $BX = 0$.

Solution:

Let P_1 be the solution space of $(AB)X = 0$, and P_2 be the solution space of $BX = 0$.
then $\dim P_1 = \dim P_2$, and it is trivial that $P_2 \subseteq P_1$ and hence $P_1 = P_2$. Q.E.D.

6. Calculate the following determinant:

$$det\begin{pmatrix} 6 & 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 6 & 2 \\ 2 & 2 & 2 & 2 & 6 \end{pmatrix}$$

Solution:

$$det\begin{pmatrix} 6 & 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 6 & 2 \\ 2 & 2 & 2 & 2 & 6 \end{pmatrix} \xrightarrow{R_1 \to R_1 + \sum_{i>1} R_i} det\begin{pmatrix} 14 & 14 & 14 & 14 & 14 \\ 2 & 6 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 6 & 2 \\ 2 & 2 & 2 & 2 & 6 \end{pmatrix}$$

$$\to 14 \cdot \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 6 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 6 & 2 \\ 2 & 2 & 2 & 2 & 6 \end{pmatrix} \xrightarrow{R_i \to R_i - 2R_1} 14 \cdot det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

And this is an upper triangular matrix, thus:

$$= 14 \cdot 4^4 = 7 \cdot 2^9.$$